ON THE FORM OF THE GENERALIZED OHM'S LAW IN A COMPLETELY IONIZED GAS

(O FORME OBOBSHCHENNOGO ZAKONA OMA V POLNOST'IU IONIZOVANNOM GAZE)

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The equations which hold for the motion of a completely ionized gas, including the equations which connect the current density with the other parameters defined by the problem, may often make use of a two-component fluid model, consisting of electrons and ions. In some cases (see, for instance, [1]) both components of the mixture and the mixture as a whole may be taken as ideal fluids. In other cases [2] in the equation of motion of the mixture one takes into account the terms connected with the viscosity both of the components and of the mixture as a whole. The present note considers the question of the calculation of the viscosity of the components in the equation used for the current density. This equation is usually called the generalized Ohm's law. Incidentally there is also obtained a non-dimensional criterion upon which depends the form of the generalized Ohm's law for a completely ionized gas.

Let the gas consist of electrons and singly charged ions. For simplicity we will assume that the number of electrons and ions in a unit volume is identical and equals n; then under certain conditions [3] the equations of the motion for each of the components can be written in the form

$$m_e n \frac{d_e \mathbf{v}_e}{dt} = -\nabla p_e - \operatorname{div} \pi_e - en\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_e \times \mathbf{H}]\right) + \mathbf{R}_e \qquad (1)$$

$$m_i n \frac{d_i \mathbf{v}_i}{dt} = -\nabla p_i - \operatorname{div} \pi_i + en\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_i \times \mathbf{H}]\right) + \mathbf{R}_i \qquad (2)$$

$$\frac{d_e}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_e \nabla, \qquad \frac{d_i}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_i \nabla$$

Here m_e, m_i are the masses of the electrons and the ions respectively ($m_e \ll m_i$), v_e, v_i are the macroscopic velocities, p_e, p_i are partial pressures, π_e, π_i are the tensors of the viscous stresses for the electrons and ion gases respectively [3], e is the value of the charge of the electron, and **E** and **H** are electric and magnetic field vectors. Interaction between the components occurs as a result of the collisions and is reduced to a certain average force $\mathbf{R}_\alpha$ ($\alpha = e,\ i$), equal to the average variation of the momentum in the collision of particles belonging to each of the different components ($\mathbf{R}_e = -\mathbf{R}_i$).

We will consider that the ion and the electron temperatures are the same ($m_e \nu_{ex}^2 \sim m_i \nu_{ix}^2$, where ν_{ex} and ν_{ix} are the velocity of random motion of the electrons and the ions). From this $p_e \sim p_i$.

If we also consider that the velocity of the relative motion of the components is small compared to the random velocities, then $p_i = p_e = p/2$ (where p is the pressure of the mixture). Multiplying (1) by e/m_e, (2) by e/m_i and adding, we obtain

$$en\left(-\frac{d_e\mathbf{v}_e}{dt} + \frac{d_i\,\mathbf{v}_i}{dt}\right) = \frac{e}{m_e}\nabla p_e + \operatorname{div}\left(\frac{e}{m_e}\pi_e - \frac{e}{m_i}\pi_i\right) +$$
$$+\frac{e^2n}{m_e}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) - \frac{e}{m_e c}\mathbf{j}\times\mathbf{H} - \frac{e^2n}{m_e}\frac{1}{\sigma}\mathbf{j} \qquad (m_e \ll m_i) \qquad (3)$$
$$\sigma = \frac{ne^2\tau_e}{m_e}, \qquad \mathbf{j} = -en(\mathbf{v}_e - \mathbf{v}_i)$$

Here σ is the conductivity of the gas in the absence of a magnetic field, $\mathbf{j}$ is the current density, and τ_e is the time between the collision of the electrons and the ions. In the case in which each of the components has a Maxwellian distribution of velocity one has

$$\mathbf{R}_e = -\frac{nm_e(\mathbf{v}_e - \mathbf{v}_i)}{\tau_e}$$

Corrections for this expression, when the distribution is close to Maxwellian, are given in [3]; but those corrections are, according to further evaluations, negligible.

Furthermore, in order to simplify the formulation we will assume that because of the greater mass of the ions the average velocity of the mixture coincides with the average velocity of the ion gas ($m_i\mathbf{v}_i \gg m_e\mathbf{v}_e$). For this case one has the relations

$$\mathbf{v} \sim \mathbf{v}_i, \qquad \mathbf{v}_e \sim \mathbf{v} - \frac{1}{en}\mathbf{j}$$

Using these relations, and also the equation of continuity

$$\frac{dn}{dt} + n\operatorname{div}\mathbf{v} = 0$$

we transform Equation (3) into the form

$$\frac{d\mathbf{j}}{dt} + \mathbf{j}\,\mathrm{div}\,\mathbf{v} + (\mathbf{j}\nabla)\mathbf{v} - (\mathbf{j}\nabla)\frac{\mathbf{j}}{en} = -\frac{e^2 n}{m_e}\frac{1}{\sigma}\mathbf{j} +$$
$$+\frac{e^2 n}{m_e}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) - \frac{e}{m_e c}\mathbf{j}\times\mathbf{H} + \frac{e}{m_e}\nabla p_e + \mathrm{div}\left(\frac{e}{m_e}\pi_e - \frac{e}{m_i}\pi_i\right) \quad (4)$$

We will consider that the characteristic time t for the problem is much greater than τ_e and τ_i the times between the collisions of ions with ions ($t \gg \max\{\tau_e, \tau_i\}$). Using this in (4) it is possible to neglect the leading term on the left-hand side in comparison with the leading term on the right-hand side.

In the problem in which the electromagnetic field significantly influences the motion, the magnetic forces are of the same order as the inértia forces.

$$\rho V^2 \sim \frac{1}{c} jHL, \quad \text{or} \quad j \sim \frac{n m_i V^2 c}{HL} \quad (\rho = n(m_e + m_i) = n m_i) \quad (5)$$

Here V, L are the characteristic velocity and length of the problem. If, in addition,viscous forces are essential in the motion, then

$$\eta\frac{V}{L} \sim \rho V^2 \quad \text{or} \quad \eta \sim n m_i V L \quad (\eta = 0.96 n T \tau_i) \quad (6)$$

Here T is the temperature, and the value of η is taken according to [3].

For the evaluation of the terms connected with the viscosity in Equation (4) it is enough to evaluate one of the components of the tensor whose divergence appears in (4), because the remaining components have the same order of magnitude [3]. The component Π_{xx} of this tensor is easily transformed into the form

$$\Pi_{xx} = \frac{e}{m_e}\pi_{exx} - \frac{e}{m_i}\pi_{ixx} = 0{,}96\,enT\left\{\frac{\tau_e}{m_e}\left[\frac{\partial w}{\partial z} - \frac{\partial}{\partial z}\frac{j_z}{en} - \right.\right.$$
$$-\frac{1}{3}\left(\mathrm{div}\,\mathbf{v} - \mathrm{div}\frac{\mathbf{j}}{en}\right) - a_2'\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial}{\partial x}\frac{j_x}{en} + \frac{\partial}{\partial y}\frac{j_y}{en}\right) -$$
$$\left.-2\omega_e\tau_e a_2''\left(\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{j_x}{en} + \frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\frac{j_y}{en}\right)\right] - \frac{\tau_i}{m_i}\left[\frac{\partial w}{\partial z} - \right. \quad (7)$$
$$\left.\left.-\frac{1}{3}\mathrm{div}\,\mathbf{v} - b_2'\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - 2\omega_i\tau_i b_2''\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\right\}$$

The coefficients a_2', a_2'', b_2' and b_2'' which appear in this expression are functions of $\omega_e\tau_e$ and $\omega_i\tau_i$ (ω_i is the Larmor ion frequency) and their order of magnitude does not exceed unity in the range of variation of $\omega_e\tau_e$ and $\omega_i\tau_i$. In connection with this

$$\frac{\tau_e}{\tau_i} \sim \sqrt{\frac{m_e}{m_i}} \ll 1 \tag{8}$$

because (see, for example, [3]) the leading terms in (7) are

$$\text{either} \quad \frac{\tau_e}{m_e}\frac{\partial w}{\partial z}, \qquad \text{or} \quad \frac{\tau_e}{m_e}\frac{\partial}{\partial z}\frac{j_z}{en}$$

Making use of the estimates in (5), (6) and (8) for terms appearing in Expression (4'), we get

$$A_1 = \mathbf{j}\,\mathrm{div}\,\mathbf{v} \sim (\mathbf{j}\nabla)\,\mathbf{v} \sim \frac{enm_iV^2cV}{eHL^2} = enV\,\frac{\Omega^2}{\omega_i}$$

$$A_2 = (\mathbf{j}\nabla)\,\frac{\mathbf{j}}{en} \sim \frac{nm_iV^2cnm_iV^2c}{HLenLHL} = enV\,\frac{\Omega^3}{\omega_i^2}$$

$$A_3 = \frac{e^2n}{m_e}\frac{1}{\sigma}\,\mathbf{j} = \frac{\mathbf{j}}{\tau_e} \sim \frac{nm_iV^2c}{HL\tau_e} = enV\,\frac{\Omega}{\omega_i\tau_e}$$

$$A_4 = \frac{e^2n}{m_e}\,\mathbf{E} \gtrsim \frac{e^2n}{m_ec}\,[\mathbf{v}\times\mathbf{H}] \sim enV\omega_e$$

$$A_5 = \frac{e}{m_e}\,\nabla p_e \lesssim \frac{e}{m_ec}\,\mathbf{j}\times\mathbf{H} \sim \frac{e}{m_ec}\,\frac{nm_iV^2c}{HL}\;H = enV\,\frac{m_i}{m_e}\,\Omega$$

$$A_6 = \frac{\partial}{\partial x}\left[0.96\,enT\,\frac{\tau_e}{m_e}\,\frac{\partial w}{\partial z}\right] \sim 0.96\,nT\tau_i\,\frac{e}{m_e}\,\frac{\tau_e}{\tau_i}\,\frac{V}{L^2}$$

$$= \eta\,\frac{e}{m_e}\,\frac{V}{L^2}\sqrt{\frac{m_e}{m_i}} \sim enV\sqrt{\frac{m_i}{m_e}}\,\Omega$$

$$A_7 = \frac{\partial}{\partial x}\left[0.96\,enT\,\frac{\tau_e}{m_e}\,\frac{\partial}{\partial z}\,\frac{j_z}{en}\right] \sim 0.96\,nT\tau_i\,\frac{\tau_e}{\tau_i}\,\frac{nm_ieV^2c}{HLenm_eL^2}$$

$$= \eta\,\frac{\tau_e}{\tau_i}\,\frac{m_i}{m_e}\,\frac{eV^2c}{eHL^3} \sim en\,V\sqrt{\frac{m_i}{m_e}}\,\frac{\Omega^2}{\omega_i}$$

$$\left(\Omega = \frac{V}{L} = \frac{1}{t},\ \frac{\Omega}{\omega_i} \sim \frac{v_e - v_i}{V}\right)$$

Here Ω is the characteristic frequency of the problem and the order of magnitude of the ratio Ω/ω_i is given on the basis of (5).

It is easy to see that the relative value of the terms which appear in (4) depends on the value of the non-dimensional parameters Ω/ω_i and $\omega_e\tau_e$.

Depending on these parameters, the generalized Ohm's law will have one or the other of the following forms:

1. The case $\Omega/\omega_i \ll 1$, $\omega_e\tau_e \ll 1$.

Comparing the terms A_1 with A_6 and A_2 with A_7 we obtain

$$\frac{A_1}{A_6} \sim \frac{A_2}{A_7} \sim \frac{\Omega}{\omega_i}\sqrt{\frac{m_e}{m_i}}$$

that is, the terms A_1 and A_2 can be neglected with respect to A_6 and A_7 for $\Omega/\omega_i \leqslant 1$. In addition, in order to satisfy the inequality, one has $A_6 \geqslant A_7$.

a) For $\Omega/\omega_i \sim \omega_e\tau_e$ we have $A_4 \gg A_5$ and $A_3 \sim A_4$. Comparing A_4 with A_6, by virtue of (8) we have

$$\frac{A_6}{A_4} \sim \sqrt{\frac{m_i}{m_e}}\frac{\Omega}{\omega_e} \sim \frac{\Omega}{\omega_i}\sqrt{\frac{m_e}{m_i}} \ll 1 \tag{9}$$

that is, the terms connected with the viscosity can be neglected. The generalized Ohm's law has the form

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right) \tag{10}$$

This form of Ohm's law can be used in magnetohydrodynamics.

b) For $\Omega/\omega_i \ll \omega_e\tau_e$ we have $A_3 \ll A_4$ and Ohm's law takes the form

$$\mathbf{E} = -\frac{1}{c}\mathbf{v}\times\mathbf{H} \tag{11}$$

This form can be used in magnetohydrodynamics for the study of the motion of an infinitely conducting medium.

c) For $\omega_e\tau_e \ll \Omega/\omega_i \ll 1$ we have $A_3 \gg A_4$ and the term A_6 must be compared with A_3; we have

$$\frac{A_6}{A_3} \sim \sqrt{\frac{m_i}{m_e}}\,\omega_i\tau_e \sim \sqrt{\frac{m_e}{m_i}}\,\omega_e\tau_e \ll 1 \tag{12}$$

that is, the terms connected with viscosity can be neglected and Ohm's law has the form

$$\mathbf{j} = \sigma\mathbf{E} \tag{13}$$

This form coincides with Ohm's law for rigid immovable conductors.

2. The case $\Omega/\omega_i \ll 1$, $\omega_e\tau_e \sim 1$. For this, the inequality $\Omega/\omega_i \ll \omega_e\tau_e$ holds and it is analogous with case 1b.

3. The case $\Omega/\omega_i \ll 1$, $\omega_e\tau_e \gg 1$ is analogous to 1b.

4. The case $\Omega/\omega_i \sim 1$, $\omega_e\tau_e \ll 1$. Here one has the relations $A_4 \sim A_5$,

$A_3 \gg A_4$, and also relation (12) holds; therefore this case is analogous to 1c.

5. The case $\Omega/\omega_i \sim 1$, $\omega_e\tau_e \sim 1$. For this case the terms A_3, A_4 and A_5 are of the same order. Comparing A_6 with A_4 we arrive at the estimate (9), and thus it follows that the terms connected with the viscosity can be neglected. Ohm's law takes on the form

$$\frac{e^2n}{m_e}\frac{1}{\sigma}\mathbf{j} + \frac{e}{m_e c}\{\mathbf{j}\times\mathbf{H} - c\nabla p_e\} - \frac{e^2n}{m_e}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) = 0 \tag{14}$$

This form of Ohm's law can be used in the problems of the motion of gas with anisotropic conductivity.

6. The case $\Omega/\omega_i \sim 1$, $\omega_e\tau_e \gg 1$. For this we have inequality (9) holding and also the inequality $A_3 \ll A_4$. Ohm's law has the form

$$\frac{1}{c}\mathbf{j}\times\mathbf{H} - \nabla p_e - en\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) = 0 \tag{15}$$

7. The case $\Omega/\omega_i \gg 1$, $\omega_e\tau_e \ll 1$. For this case $A_2 \gg A_1$, $A_7 \gg A_6$, $A_5 \ll A_3$, $A_5 \gg A_4$. Comparing A_7 and A_3, we get

$$\frac{A_7}{A_3} \sim \sqrt{\frac{m_i}{m_e}}\,\Omega\tau_e \sim \frac{\tau_i}{t} \ll 1 \tag{16}$$

that is, the terms connected with the viscosity can be neglected.

Comparing A_2 and A_7 we obtain

$$\frac{A_2}{A_7} \sim \frac{\Omega}{\omega_i}\sqrt{\frac{m_e}{m_i}}$$

It follows that for $1 \ll \Omega/\omega_i \lesssim \sqrt{(m_i/m_e)}$ one has $A_2 \leqslant A_7 \leqslant A_3$ and Ohm's law has the form (13). If $\Omega/\omega_i > \sqrt{(m_i/m_e)}$, then $A_3 \sim A_2$ and Ohm's law has the form

$$\mathbf{j} = \sigma\mathbf{E} + \tau_e(\mathbf{j}\nabla)\frac{\mathbf{j}}{en} \tag{17}$$

8. The case $\Omega/\omega_i \gg 1$, $\omega_e\tau_e \sim 1$. For this case $A_3 \sim A_5$, $A_2 \gg A_1$, $A_7 \gg A_6$, $A_5 \gg A_4$. Comparing A_7 with A_3, using relation (16) and comparing A_2 with A_5, we obtain

$$\frac{A_2}{A_5} \sim \frac{\Omega^2}{\omega_i^2}\frac{m_e}{m_i} \sim \frac{\Omega^2\omega_i^2\tau_i^2}{\omega_i^2\omega_e^2\tau_e^2} \sim \frac{\tau_i^2}{t^2} \ll 1 \tag{18}$$

It follows that Ohm's law has the form

$$\mathbf{j}+\frac{\tau_e e}{m_e c}\{\mathbf{j}\times\mathbf{H}-c\nabla p_e\}=\sigma\mathbf{E} \tag{19}$$

9. The case $\Omega/\omega_i \gg 1$, $\omega_e\tau_e \gg 1$. For this case the relations (16) and (18) hold and also $A_3 \ll A_5$, and it follows that Ohm's law has the form

$$\frac{1}{c}\mathbf{j}\times\mathbf{H}-\nabla p_e-\mathrm{en}\,\mathbf{E}=0 \tag{20}$$

If

$$|\mathbf{E}|\sim\frac{1}{c}|\mathbf{v}\times\mathbf{H}|$$

then, in Equations (19) and (20), the terms containing $\mathbf{E}$ can be omitted.

In such a manner, the terms connected with the viscosity can always be neglected in obtaining Ohm's laws for a completely ionized gas within the two-component model. If relation (6) is invalid, that is, the viscous stresses can be neglected in the equations of motion, then all of the estimates connected with viscous terms in Ohm's law are strengthened and again those terms in Ohm's law can, of course, be neglected.

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